Basic Inequalities

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Cauchy's Inequality (also called Cauchy-Schwarz, Cauchy-Buniakowski, etc.) states that for positive reals $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$,

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2)$$

or

$$\left(\sum x_iy_i\right)^2 \leq \left(\sum x_1^2\right)\left(\sum y_1^2\right)$$

with equality iff $x_1/y_1 = x_2/y_2 = \cdots = x_n/y_n$.

For positive reals $a_1, a_2, \ldots, a_n$, the Root Mean Square - Arithmetic Mean - Geometric Mean - Harmonic Mean or **RMS-AM-GM-HM** Inequality states that

$$\frac{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}}{n} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1a_2\cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

with equality iff $a_1 = a_2 = \cdots = a_n$.

Problems

1 (IMO 1995/2). Let a, b, c be positive reals with $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solutions

1 (Solution 1). By Cauchy,

$$\left(\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)}\right)(a(b+c) + b(c+a) + c(a+b)) \geq \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)^2.$$

But the LHS is equivalent to

$$2\left(\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)}\right)(ab + bc + ca).$$

Applying GM-HM on the RHS yields

$$\frac{3}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} \le \sqrt[3]{a^2b^2c^2} \iff \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \ge 3.$$

Also by GM-HM,

$$\sqrt[3]{abc} \ge \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = \frac{3}{ab+bc+ca} \iff ab+bc+ca \ge 3.$$

Thus

$$\left(\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}\right) \ge \frac{3^2}{3\cdot 2} = \frac{3}{2},$$

as desired. □

1 (Solution 2). [*Thanks to Ercole Suppa for correcting an error in this solution.*] Before presenting this solution we introduce **cyclic sums**, denoted by $\sum_\sigma f(x_1, x_2, \ldots, x_n)$.

$$\sum_\sigma f(x_1,x_2,\ldots,x_n) = f(x_1,x_2,\ldots,x_n)+f(x_2,x_3,\ldots,x_1)+f(x_3,x_4,\ldots,x_2)+\cdots+f(x_n,x_1,\ldots,x_{n-1}),$$

i.e. we apply the transformation $(x_1, x_2, \ldots, x_n) \to (x_2, x_3, \ldots, x_1)$ on each term to obtain the next term in the sum.

Since $abc = 1$ we have $\frac{1}{a^3} = (bc)^3$ so we must show that

$$\sum_\sigma \frac{(bc)^3}{b+c} \ge \frac{3}{2}.$$

By AM-GM (each term is positive and real),

$$\sum_\sigma \frac{(bc)^3}{b+c} \ge \frac{3}{\sqrt[3]{(b+c)(c+a)(a+b)}}.$$

Thus it is sufficient to show that $\sqrt[3]{(b+c)(c+a)(a+b)} \ge 2$. By AM-GM, $b+c \ge 2\sqrt{bc}$, $c+a \ge 2\sqrt{ca}$, and $a+b \ge 2\sqrt{ab}$. If we multiply these inequalities together, we get $(b+c)(c+a)(a+b) \ge 8\sqrt{a^2b^2c^2} = 8$, and the desired inequality follows. □

Practice Problems

1. (ARML 1987) If a, b and c are each positive and $a+b+c=6$, show that

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \ge \frac{75}{4}.$$

2. Prove that for any integer $n > 1$

$$n! < \left(\frac{n+1}{2}\right)^n.$$

3. Prove that if α, β, γ are the three angles of a triangle, then

$$\tan^2 \frac{\alpha}{2} + \tan^2 \frac{\beta}{2} + \tan^2 \frac{\gamma}{2} \geq 1.$$

4. (USSR Olympiad Problem Book) Verify that for any three arbitrary numbers x_1, x_2, x_3 the following inequality holds:

$$\left(\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{6}x_3\right)^2 \leq \frac{1}{2}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{6}x_3^2.$$